Preference functions and thresholds

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June 14, 2018

Introduction

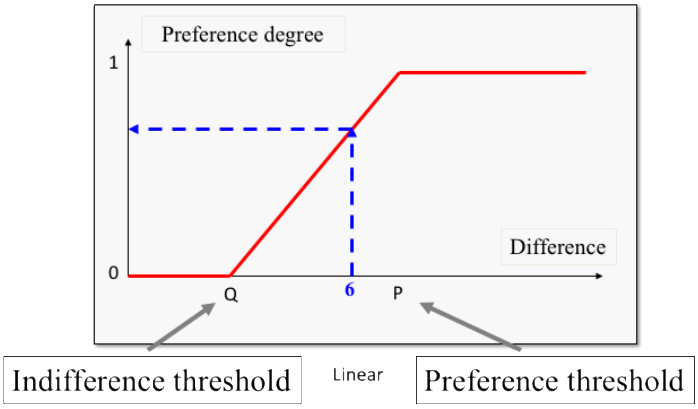
The **PROMETHEE** methods belong to the family of the outranking methods for multicriteria decision aid. As such, they are based on modeling the pairwise comparisons of the actions (decisions, alternatives, …).

The pairwise comparisons model of **PROMETHEE** requires that a preference function be associated to each criterion. The purpose of the preference function is to translate the difference observed between two actions on a given criterion, from the criterion scale to a normalized 0-1 degree of preference.

Preference function

A specific preference function is associated separately to each criterion in a multicriteria decision problem. We thus hereafter consider the case of a single criterion.

The following figure shows the shape of a typical preference function (type V – linear preference function).



The horizontal axis corresponds to the value of the difference between two actions on one single criterion. It is important to remember that it is the difference between two values that matters here and not absolute values: preference functions are used to compare two actions.

The vertical axis corresponds to a degree of preference between two actions: a value of 0 means no preference at all, while a value of 1 means absolute undisputable preference for the better action.

Logically, the preference function is a non-decreasing function of the difference: larger differences cannot lead to lower preference degrees.

Let us take an example as illustration. We consider the problem of purchasing a new laptop computer. One criterion is the price of the laptop (expressed in €). The preference function

associated to this criterion models the way we compare the prices of two laptops. Several situations are possible:

1. If laptop A costs 799€ and laptop B costs 1499€, the price difference is 700€. It is a lot of money. And it should translate to a full undisputable preference (preference degree = 1) in favor of the cheaper laptop A.
2. If laptop A costs 799€ and laptop B costs 819€, the price difference is 20€. For most people, this difference is negligible, especially as other criteria are taken into account. The preference degree is then equal to 0. A and B are said to be indifferent with respect to price.
3. If laptop A costs 799€ and laptop B costs 859€, the price difference is 60€. This is certainly not negligible, the higher price of B should be taken into account, but it also most probably can be compensated for by better performance of B on some other criteria. There is some kind of hesitation. The preference for A over B is not so clearly established, it is weaker. In that case the degree of preference for A over B is positive but less than 1. There is some preference for A over B but it is not a full preference.

Most preference function shapes include one or two threshold values. For instance, the linear preference function represented in the previous figure includes two thresholds:

- Q is the indifference threshold: it is the largest difference that is considered as negligible (preference degree = 0).
- P is the preference threshold: it is the smallest difference that corresponds to a full undisputable preference (preference degree = 1).

In our illustrative example:

- Q should be larger than 20€.
- P should be smaller than 700€ and larger than 60€.

Types of preference function

PROMETHEE, and most associated software, proposes the six following types (shapes) of preference functions. Additional shapes could be considered as well but, in practice, these six shapes have been found to provide decision makers with a sufficient number of options to accommodate most typical situations.

Basic preference functions

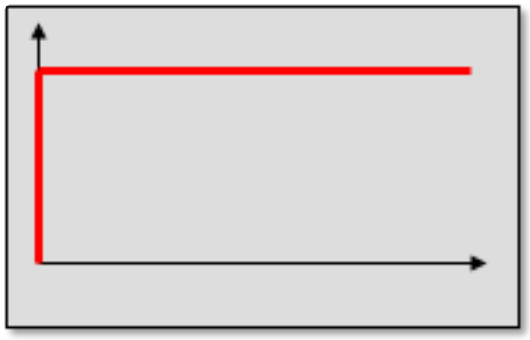
The first three preference function types introduce basic preference modelling notions.

Type I – Usual preference function

This is the simplest of all preference functions. It has no thresholds and returns a binary result:

- Two actions with equal values (difference = 0) are indifferent (preference degree = 0).
- Two actions with different values (difference > 0) generate a full preference (preference degree = 1) even if the difference is very small.

This type of preference function should be used very carefully as it doesn't distinguish between very small differences, that could be negligible, and much larger ones.

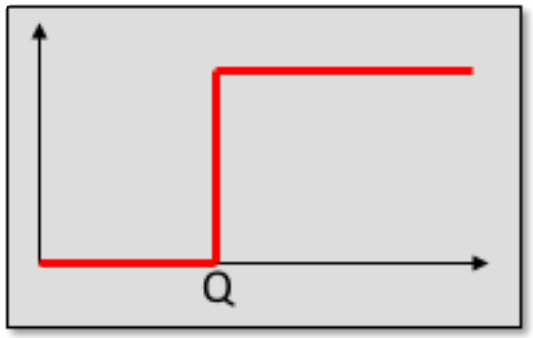
 Usual	**Thresholds:** None.

Type II – U-shape preference function

The U-shape preference function introduces the notion of indifference threshold (Q) but is still binary:

- Two actions with close values (difference <= Q) are indifferent (preference degree = 0).
- Two actions with more different values (difference > Q) generate a full preference (preference degree = 1).

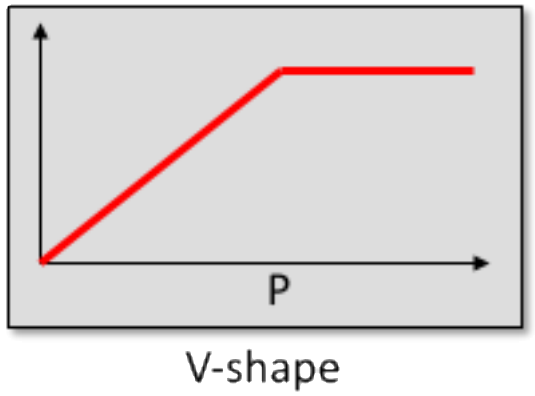
Its range of use is quite limited in practice.

 U-shape	**Thresholds:** Q – indifference threshold

Type III – V-shape preference function

The V-shape preference function introduces the notion of preference threshold (P) and of variable preference degree:

- Two actions with equal values (difference = 0) are indifferent (preference degree = 0).
- Two actions with quite different values (difference > P) generate a full preference (preference degree = 1).
- Two actions with smaller different values (difference <= P) generate a preference degree proportional to the difference (preference degree = difference / P).

 V-shape	**Thresholds:** P – preference threshold

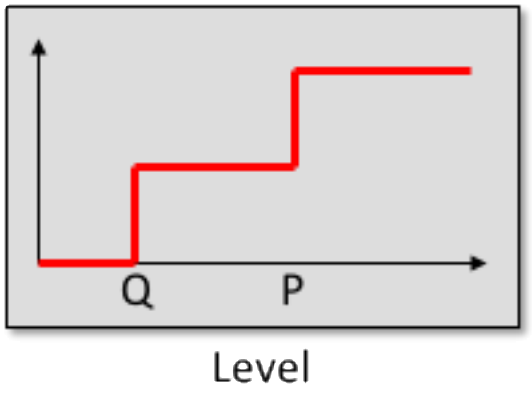
Advanced preference functions

The last three preference function types allow for improved preference modelling.
The Q and P thresholds have been introduced separately in the previously defined preference functions. However, both thresholds bring different elements of preference modelling and should often be integrated together in preference functions. This leads to the type IV (Level) and V (Linear) preference functions.
The type VI (Gaussian) preference function has been proposed as an alternative to the type V (Linear) preference function.

Type IV – Level preference function

The Level preference function includes two thresholds: Q and P. According to the definition of theses thresholds, three distinct cases appear:

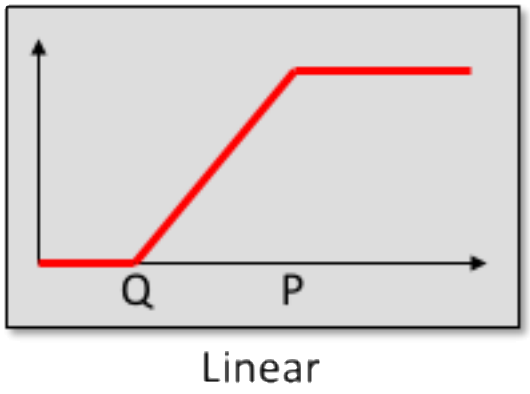
- Two actions with very close values (difference <= Q) are indifferent (preference degree = 0).
- Two actions with quite different values (difference > P) generate a full preference (preference degree = 1).
- In between, two actions with different values (Q < difference <= P) generate a weak preference degree (preference degree = 1/2).

 Level	**Thresholds:** Q – indifference threshold P – preference threshold

Type V – Linear preference function

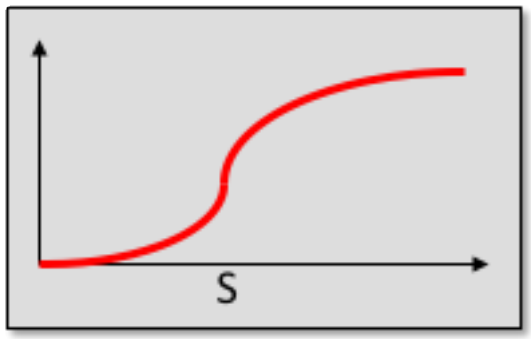
The Linear preference function also includes two thresholds: Q and P. The main difference with Type IV (Level) preference function is that the degree of preference is linearly increasing between the Q and P thresholds:

- Two actions with very close values (difference <= Q) are indifferent (preference degree = 0).
- Two actions with quite different values (difference > P) generate a full preference (preference degree = 1).
- In between, two actions with different values (Q < difference < P) generate a preference degree that is linearly increasing from 0 to 1 as the difference is increasing from Q to P (preference degree = (difference - Q) / (P - Q)).

 Linear	**Thresholds:** Q – indifference threshold P – preference threshold

Type VI – Gaussian preference function

The Gaussian preference function has been designed as an alternative to the Linear (type V) preference function. It has a smoother shape without flat indifference and full preference areas. At the same time, it doesn't have any indifference or preference thresholds. Its shape is determined by another parameter: the S Gaussian threshold which defines the position of the inflexion point of the preference function curve. Practically, the value of the S threshold should be in between the values of the Q and P thresholds, as it corresponds to a preference degree of 0.39). It is however more difficult to define than the Q and P thresholds.

 Gaussian	**Thresholds:** S – Gaussian threshold

Comparing preference functions

While there are six different types of preference functions in the **PROMETHEE** methodology, some are actually special cases of others and were introduced mostly for didactic purpose. We can identify three families of preference functions according to their specific shapes:

1. The “step” shape represented by the Type IV Level preference function:
    - The Type I Usual preference function is a special case with Q = P = 0
    - The Type II U-shape preference function is a special case with Q = P > 0
2. The “linear” shape represented by the Type V Linear preference function:
    - The Type III V-shape preference function is a special case with Q = 0
3. The “Gaussian” shape represented by the Type VI Gaussian preference function.

Choosing an adequate type of preference function

Taking into account the scale of measurement of the criterion

The choice of a good preference function depends on the scale of the underlying criterion.

To keep it simple, let us consider three different cases:

1. A continuous numerical scale (real numbers):
    a. Example: the price of the laptop. It is measured in € (or any other currency) and any (positive) value is possible, including non-integer ones (down to 1 cent precision for this example).
    b. Many different values are possible. Some can be very close to each other (789€ vs 789.5€). Some can be quite different from each other (789€ vs 1089€).
    c. In this case it is important to distinguish small, probably negligible, differences from larger ones and to avoid “step” effects that can appear with the “step” shape (a very small variation of the difference can result in a large variation of the preference degree).
2. A discrete numerical scale (small integer numbers):
    a. Example: the number of USB-C ports available on the laptop. This can be 0, 1, 2, 3, 4, … but certainly not 10 or more!
    b. A limited number of distinct values are possible. Most values are quite different from each other (e.g. one additional USB-C port is already a big improvement).
    c. In this case, the different possible values are relatively different from each other and the “step” shape can be considered as a good alternative.
3. A qualitative scale (few ordered levels):
    a. Example: the quality of the laptop. This can be measured on a 5-point Likert scale (Very good, Good, Average, Bad, Very bad).
    b. A limited number of levels are possible (usually less than 10)
    c. In this case, the different possible values are relatively different from each other and the “step” shape can be considered as a good alternative.

A simple procedure

Here is simple step-by-step procedure for selecting a preference function.

1. If the criterion has a continuous numerical scale:

   a. If you want to introduce an indifference threshold (and thus neglect very small differences), consider using a Type V Linear preference function.
   b. If you want that even very small differences play some role in the **PROMETHEE** computation, consider using a Type III V-shape preference function.
2. If the criterion has a discrete numerical scale or a qualitative scale:
   a. If the number of possible values is small (<= 5) and if the values are perceived as quite different from each other, consider using a Type I Usual preference function.
   b. If the number of possible values is larger or if you want to have a weaker degree of preference for smaller differences, consider using a Type IV Level preference function.

This procedure doesn't include the Type II (U-shape) and Type VI (Gaussian) preference functions. In practice, the Type II preference function is seldom used. The Type VI preference function can be used as an alternative to the Type V preference function.

Determination of the associated thresholds

Once the type of preference function has been selected, the values of the corresponding threshold(s) have to be determined according to the perception of the criterion scale by the decision-maker.

Depending of the type of preference function, 0, 1 or 2 thresholds have to be determined:

- For a Type I Usual preference function, no threshold has to be determined.
- For all other types of preference function, 1 or 2 thresholds (Q, P or S) have to be determined.

Remember that thresholds correspond to differences and not to absolute values.

Determination of a Q indifference threshold

The Q indifference threshold is the largest difference between two actions that doesn't generate any degree of preference.
Let us come back to the example of the price of a laptop:

1. If laptop A costs 789€ and laptop B costs 795€, the price difference is 6€. For most people, this will be negligible as it is small and can be much compensated by differences on other criteria. With respect to price, A and B are indifferent. Thus, Q is not smaller than 6€.
2. If laptop A costs 789€ and laptop B costs 829€, the price difference is 40€. It is not a lot of money yet, but it makes for some degree of preference in favor of A. Which means that Q is smaller than 40€.
3. By considering different price differences (larger than 6€, smaller than 40€), it is possible to get closer to the value of Q. In this example, the value of Q could be close to, let us say, 25€.

In practice, the value of the Q threshold can be obtained by successive approximations, starting with very small differences (lower than Q) and progressively increasing the differences until the decision maker feels the difference is not anymore negligible.
It is not important to obtain a precise value for Q. However, the magnitude of Q is important. In our laptop price example, Q could be 25€, or 20€, or 30€. These different values will have some impact on the **PROMETHEE** computations, but the impact will be

marginal. However, if Q is set to 250€, one can expect inconsistent results as price differences up to 250€ will be considered as negligible.

Determination of a P preference threshold

The P preference threshold is the smallest difference between two actions that leads to a full degree of preference. It is thus significantly larger than the Q threshold.
Let us again come back to the example of the price of a laptop:

1. If laptop A costs 789€ and laptop B costs 1289€, the price difference is 500€. For most people, this is a huge difference that is difficult to compensate by better performance on other criteria. With respect to price, there will be a full preference for A. Thus, P is smaller than 500€.
2. If laptop A costs 789€ and laptop B costs 829€, the price difference is 40€. It is not a lot of money, but it is sufficient to make for some lower degree of preference in favor of A. Which means that P is larger than 40€.
3. By considering different price differences (smaller than 500€, larger than 40€), it is possible to get closer to the value of P. In this example, the value of P could be close to, let us say, 100€.

In practice, the value of the P threshold can be obtained by successive approximations, starting with larger differences (larger than P) and progressively decreasing the differences until the decision maker feels the difference is not a full one.
As for Q, it is not important to obtain a precise value for P. However, the magnitude of P is important. In our laptop price example, P could be 100€, or 90€, or 125€. These different values will have some impact on the **PROMETHEE** computations, but the impact will be marginal. However, if P is set to 10€, one can expect inconsistent results as smaller price differences as low as 10€ will be considered as important as much larger differences.

Determination of a S Gaussian threshold

The determination of the S threshold is trickier as it corresponds to a difference for which the preference degree is close to 0.5 (actually, 0.39). It corresponds to a “weak” preference degree. The value for S should thus be larger than a Q value and smaller than a P value.
One way to assess the value of S is to start by assessing both Q and P values and then to set S equal to their average ($S = (Q + P) /2$).

Alternate statistical approach

According to the **PROMETHEE** methodology, the values of the thresholds should be determined according to the perception of the decision maker.
Sometimes this information is difficult to obtain because the decision maker is not available or because the decision maker is not familiar with the data.
In such a case, the values of the thresholds can be based on the inspection of the “statistical” distribution of the values observed for the different actions.
The Q threshold can then be determined in such a way that smaller observed differences are negligible. Or alternatively, in a more prudent approach, the Q threshold can be set equal to 0, to avoid neglecting some useful information.
The P threshold can be set to a value that is smaller than the spread of the observed values (maximum value – minimum value). Indeed, larger values of P will result in lower preference degrees and will thus reduce the impact of the criterion in the **PROMETHEE** computations.
However, this approach doesn’t take into account the actual meaning of the numbers and it has to be used very carefully. The values obtained for the thresholds should certainly be checked for their consistency.

Let us come back to the example of the laptop price to illustrate a potential shortcoming. Let us suppose that we compare 10 different laptops. The cheapest one costs 789€ and the most expensive one costs 799€. The spread is thus equal to 10€. A pure statistical approach would lead us to set the P threshold to a value lower than 10€, which is of course much too low and will exaggerate the impact of negligible price differences.