① $f(x) = \left|\cos \frac{x}{2}\right|$

f парна $\Rightarrow b_n = 0 \quad (\forall n \in \mathbb{N})$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} \cos\frac{x}{2}\,dx = \frac{2}{\pi}\cdot \frac{\sin\frac{x}{2}}{\frac{1}{2}}\Big|_0^{\pi} = \frac{4}{\pi}.$$

$n \geq 1$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \cos\frac{x}{2}\cdot\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi}\frac{1}{2}\left[\cos\left(\frac{x}{2}+nx\right)+\cos\left(\frac{x}{2}-nx\right)\right]dx$$

$$= \frac{1}{\pi}\int_0^{\pi}\left[\cos x\left(\frac{1}{2}+n\right)+\cos x\left(\frac{1}{2}-n\right)\right]dx = \frac{1}{\pi}\left(\frac{\sin x\left(\frac{1+2n}{2}\right)}{\frac{1+2n}{2}}\Big|_0^{\pi} + \right.$$

$$\left. + \frac{\sin x\left(\frac{1-2n}{2}\right)}{\frac{1-2n}{2}}\Big|_0^{\pi}\right) = \frac{1}{\pi}\left(\frac{2}{1+2n}\sin\left(\pi\left(\frac{1+2n}{2}\right)\right) + \frac{2}{1-2n}\sin\left(\pi\,\frac{1-2n}{2}\right)\right)$$

$$= \frac{1}{\pi}\left(\frac{2}{1+2n}\sin\left(\frac{\pi}{2}+n\pi\right) + \frac{2}{1-2n}\sin\left(\frac{\pi}{2}-n\pi\right)\right)$$

$$= \frac{1}{\pi}\left(\frac{2}{1+2n}\cos n\pi + \frac{2}{1-2n}\cos n\pi\right) = \frac{1}{\pi}\left(\frac{2(-1)^n}{1+2n} + \frac{2(-1)^n}{1-2n}\right)$$

$$= \frac{2(-1)^n}{\pi}\left(\frac{1\cancel{-2n}+1\cancel{+2n}}{1-4n^2}\right) = \frac{4(-1)^n}{\pi(1-4n^2)}.$$

$$\varphi(x) = \frac{2}{\pi} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{1-4n^2}\cos nx.$$

f је непр. у $x=0 \Rightarrow f(0) = \varphi(0)$.

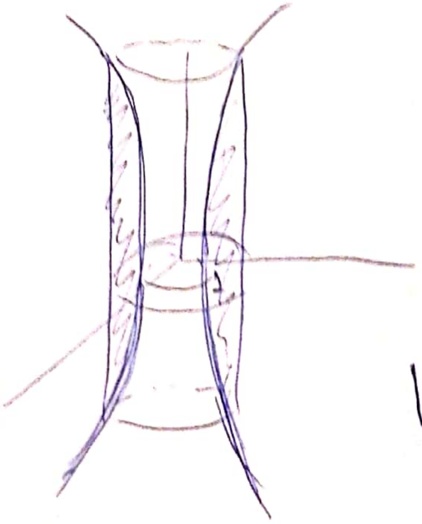
$\cos 0 = 1$

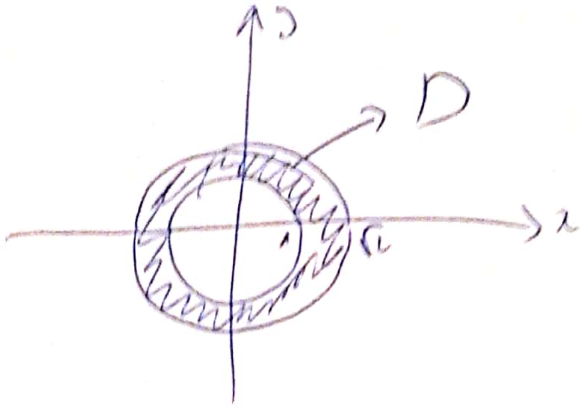
$$f(0) = \frac{2}{\pi} + \frac{4}{\pi}\cdot\sum_{n=1}^{\infty}\frac{(-1)^n}{1-4n^2} \Rightarrow \boxed{\sum_{n=1}^{\infty}\frac{(-1)^n}{1-4n^2} = \left(1-\frac{2}{\pi}\right)\cdot\frac{\pi}{4} = \frac{\pi}{4}-\frac{1}{2}}$$

2

$S_1: z^2 = x^2+y^2-1$

$S_2: x^2+y^2=2$





$$V(T) = 2\iint_D \sqrt{x^2+y^2-1}\,dx\,dy$$

$$\left[\begin{array}{lll} x=\rho\cos\varphi & J=\rho & 1\le\rho\le\sqrt{2} \\ y=\rho\sin\varphi & & -\pi\le\varphi\le\pi \end{array}\right]$$

$$V(T) = 2\int_{-\pi}^{\pi}\left(\int_1^{\sqrt{2}}\sqrt{\rho^2-1}\,\rho\,d\rho\right)d\varphi = (*)$$

$$I = \int_1^{\sqrt{2}}\sqrt{\rho^2-1}\,\rho\,d\rho = \left(\begin{array}{ll} \rho^2-1=t & \begin{array}{c|c|c} \rho & 1 & \sqrt{2} \\ \hline t & 0 & 1 \end{array} \\ 2\rho\,d\rho = dt & \\ \rho\,d\rho = \frac{1}{2}dt & \end{array}\right)$$

$$= \frac{1}{2}\int_0^1 \sqrt{t}\,dt = \frac{1}{\cancel{2}}\cdot\frac{t^{3/2}}{\frac{3}{\cancel{2}}}\Big|_0^1 = \frac{1}{3}$$

$$\Rightarrow (*)\ 2\int_{-\pi}^{\pi}\frac{1}{3}\,d\varphi = \frac{2}{3}\cdot 2\pi = \boxed{\frac{4\pi}{3}}$$

3 $\iint\limits_D \frac{\operatorname{arctg}\frac{y}{x}}{x^2}\,dx\,dy$, $D: (x-1)^2+y^2 \le \frac{1}{4}$

Приметимо да је $f(x,-y) = \frac{\operatorname{arctg}\left(-\frac{y}{x}\right)}{x^2}$

$= -\frac{\operatorname{arctg}\frac{y}{x}}{x^2} = -f(x,y)$, односно ф-ја је непарна по y. Област D је симетрична по y:

$-\sqrt{\frac{1}{4}-(x-1)^2} \le y \le \sqrt{\frac{1}{4}-(x-1)^2} \Rightarrow \iint\limits_D f(x,y)\,dx\,dy = 0.$

④ $f(x) = x \operatorname{arctg}\left(\frac{2-x}{2+x}\right) + \ln(4+x^2)$

$$f'(x) = \operatorname{arctg}\left(\frac{2-x}{2+x}\right) + x \cdot \frac{1}{\left(\frac{2-x}{2+x}\right)^2+1} \cdot \left(\frac{2-x}{x+2}\right)' + \frac{1}{4+x^2} \cdot 2x$$

$$= \operatorname{arctg}\left(\frac{2-x}{2+x}\right) + x \cdot \frac{\cancel{(x+2)^2}}{(2-x)^2+(x+2)^2} \cdot \frac{-1\cdot(x+2)-1(2-x)}{\cancel{(x+2)^2}} + \frac{2x}{4+x^2}$$

$$= \operatorname{arctg}\left(\frac{2-x}{2+x}\right) \cancel{- \frac{4x}{2x^2+8}} + \cancel{\frac{2x}{4+x^2}} = \operatorname{arctg}\left(\frac{2-x}{2+x}\right)$$

$$f''(x) = -\frac{2}{4+x^2} = -2 \cdot \frac{1}{4\left(1+\frac{x^2}{2}\right)} = -\frac{1}{2}\left(1+\frac{x^2}{4}\right)^{-1}$$

$$= -\frac{1}{2}\sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{4^n} \quad \text{za } \left|\frac{x^2}{4}\right| < 1 \Leftrightarrow |x| < 2$$

$$\Rightarrow f'(x) = -\frac{1}{2}\sum_{n=0}^{\infty} \int \frac{(-1)^n}{4^n} x^{2n}\,dx + C_1 = -\frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n(2n+1)} x^{2n+1} + C_1$$

$$\left.\begin{array}{l} f'(0) = C_1 \\ f'(0) = \operatorname{arctg} 1 \end{array}\right\} \Rightarrow C_1 = \frac{\pi}{4}$$

$$f(x) = \int \left(\frac{\pi}{4} - \frac{1}{2}\sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{4^n(2n+1)}\right) dx + C_2$$

$$= \frac{\pi}{4}x - \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{4^n(2n+1)(2n+2)} + C_2$$

$$\left.\begin{array}{l} f(0) = C_2 \\ f(0) = \ln 4 \end{array}\right\} \Rightarrow C_2 = \ln 4$$

$$\Rightarrow f(x) = \ln 4 + \frac{\pi}{4}x - \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{4^n (2n+1)(2n+2)}$$, за $x \in (-2,2)$ апсолутно.

За ✓ тачку $x=-2$ ф-ја није дефинисана.

за $x=2$

$$\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+2}}{4^n (2n+2)(2n+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n 4\cdot \cancel{4^n}}{\cancel{4^n}(2n+2)(2n+1)} = \sum_{n=0}^{\infty} \frac{4(-1)^n}{(2n+2)(2n+1)}$$

$$|a_n| \sim \frac{1}{n^2} \Rightarrow$$ ред апсолутно конв.

ф-ја је непрекидна у тачки $x=2$

$\Rightarrow$ развој важи за $x \in (-2,2]$.

Сума

$$f(1) = \ln 4 + \frac{\pi}{4} - \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n (2n+1)(2n+2)} = \ln 4 + \frac{\pi}{4} - \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n 2(n+1)(2n+1)}$$

$$= \ln 4 + \frac{\pi}{4} - \frac{1}{4} S \Rightarrow S = 4\left(\ln 4 + \frac{\pi}{4} - f(1)\right)$$

$$\Rightarrow \boxed{S = 4\left(\ln 4 + \frac{\pi}{4} - \left(\operatorname{arctg}\frac{1}{3} + \ln 5\right)\right)}$$

5. $L(y) = (x^2+3x+1)y'' + (x^2+x-2)y' - (2x+3)y = 0$

a) $L(y) = 0$

$y_1 = e^{ax}$ пар. решење.

$y_1' = ae^{ax}, \quad y_1'' = a^2 e^{ax}$

$$(x^2+3x+1)a^2e^{ax} + (x^2+x-2)ae^{ax} - (2x+3)e^{ax} = 0 \quad /: e^{ax}$$

$$a^2(x^2+3x+1) + a(x^2+x-2) - (2x+3) = 0$$

$$x^2(a^2+a) + x(3a^2+a-2) + a^2-2a-3 = 0$$

$a(a+1)=0$ $\qquad 3a^2+a-2=0 \qquad a^2-2a-3=0$

$a=0$ или $a=-1$ $\qquad$ ✓ $\qquad$ ✓

~~$a=0$ ...~~

$\Rightarrow \boxed{y_1 = e^{-x} \text{ је једно парт. решење.}}$

Друго пар. решење које је линеарно независно од y_1 добијамо из формуле

$$y_2 = y_1 \int \frac{e^{-\int p(x)dx}}{y_1^2}dx, \quad \text{где је} \quad p(x) = \frac{x^2+x-2}{x^2+3x+1}.$$

$$(*) \int p(x)dx = \int \frac{x^2+x-2}{x^2+3x+1}dx = \int \frac{x^2+3x+1-2x-3}{x^2+3x+1}dx$$

$$= \int \left(1 - \frac{2x+3}{x^2+3x+1}\right)dx = x - \int \frac{2x+3}{x^2+3x+1}dx = x - \ln(x^2+3x+1)$$

$$(**)\ e^{-\int p(x)dx} = e^{-x+\ln(x^2+3x+1)} = e^{-x}\cdot(x^2+3x+1).$$

Из $(*)$ и $(**) \Rightarrow y_2 = e^{-x}\int \frac{e^{-x}(x^2+3x+1)}{e^{-2x}}\,dx = e^{-x}\underbrace{\int e^{x}(x^2+3x+1)\,dx}_{I} = ◆$

$$I = \int e^{x}(x^2+3x+1)\,dx = \left(\begin{matrix} u = x^2+3x+1 & v = e^{x} \\ du = (2x+3)\,dx & \end{matrix}\right)$$

$$= e^{x}(x^2+3x+1) - \int e^{x}(2x+3)\,dx = \left(\begin{matrix} 2x+3 = u & v = e^{x} \\ du = 2\,dx & \end{matrix}\right)$$

$$= e^{x}(x^2+3x+1) - \left(e^{x}(2x+3) - 2\int e^{x}\,dx\right)$$

$$= e^{x}(x^2+3x+1) - e^{x}(2x+3) + 2e^{x} = e^{x}(x^2+3x+1-2x-3+2)$$

$$= e^{x}(x^2+x)$$

$$\Rightarrow ◆ = e^{-x}\cdot e^{x}(x^2+x) = x^2+x$$

$$\Rightarrow \boxed{y_2 = x^2+x}$$

Опште решење $L(y)=0$ је $\boxed{y_h = C_1 e^{-x} + C_2(x^2+x)}$

2) $L(y) = \dfrac{(x^2+3x+1)^2}{x^2+x}$

$\Rightarrow f(x) = \dfrac{x^2+3x+1}{x^2+x} \left(= 1 + \dfrac{2x+1}{x^2+x}\right)$

$C_1'(x) \cdot y_1 + C_2' y_2 = 0$

$C_1' y_1' + C_2' y_2' = f(x)$

$C_1' e^{-x} + C_2'(x^2+x) = 0$

$-C_1' e^{-x} + (2x+1) C_2' = \dfrac{x^2+3x+1}{x^2+x}$ $+$

$C_2'(x^2+3x+1) = \dfrac{x^2+3x+1}{x^2+x} \Rightarrow C_2' = \dfrac{1}{x^2+x}$

$C_2 = \displaystyle\int \frac{dx}{x(x+1)} + B = \int \left(\frac{1}{x} - \frac{1}{x+1}\right) dx + B$

$= \ln x - \ln(x+1) + B = \ln \dfrac{x}{x+1} + B$

$C_1' e^{-x} = -C_2'(x^2+x) = -1$

$\Rightarrow C_1' = -e^x \Rightarrow \boxed{C_1 = -e^x + A}$

O.P. $\boxed{y = e^{-x}(-e^x + A) + (x^2+x)\left(\ln \frac{x}{x+1} + B\right)}$