1 Gustina verovatnoće slučajne promenljive X data je izrazom:

$$f(x) = e^{-x}$$

koji je definisan za $x \geq 0$.
Slučajna promenljiva Y je sa slučajnom promenljivom X povezana relacijom

$$y = 2x = g(x)$$

Naći izraz za gustinu verovatnoće slučajne promenljive Y i grafički predstaviti obe gustine verovatnoće.

Primenićemo izraz za transformaciju gustine verovatnoće

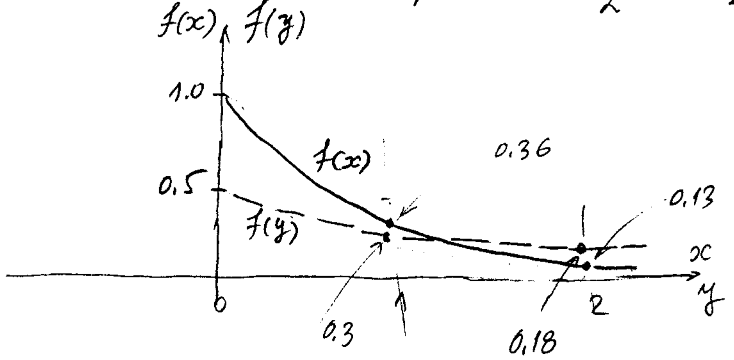
$$f_Y(y) = \left.\frac{f_X(x)}{\left|\frac{dy}{dx}\right|}\right|_{x = g^{-1}(y)}$$

$$y = 2x, \quad x = \frac{1}{2}y,$$

$$\frac{dy}{dx} = 2$$

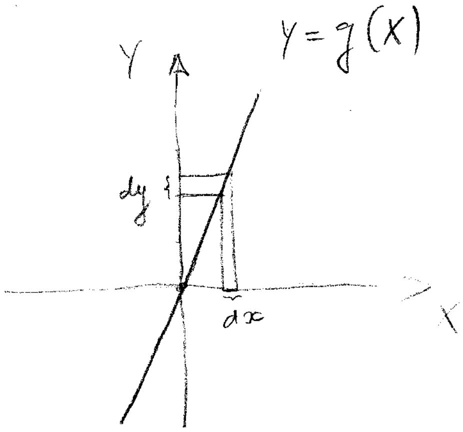
Zamenom se dobija

$$f_Y(y) = \frac{e^{-\frac{y}{2}}}{2} = \frac{1}{2}e^{-y/2}$$



$$f_Y(y)\,|dy| = f_X(x)\,|dx|$$

$$f_Y(y) = \left.\frac{f_X(x)}{\left|\frac{dy}{dx}\right|}\right|_{x = g^{-1}(y)}$$

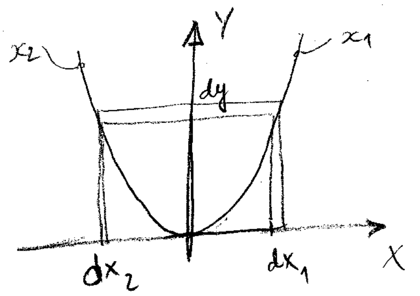


$$f_Y(y) = f_X(x) \cdot \left|\frac{dx}{dy}\right|$$

$g^{-1}(y)$

2 Slučajna promenljiva X ima Gausovu raspodelu sa nultom srednjom vrednošću i varijansom σ_x^2. Odrediti raspodelu slučajne promenljive Y za koju važi $Y = X^2$.

Rešenje



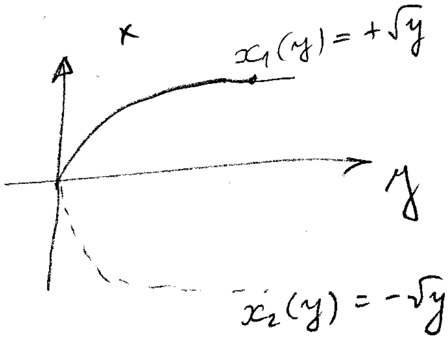
$$f_Y(y)|dy| = f_X(x_1)\cdot|dx_1| + f_X(x_2)\cdot|dx_2|$$

$$f_Y(y) = f_X(x_1)\left|\frac{dx_1}{dy}\right| + f_X(x_2)\left|\frac{dx_2}{dy}\right|$$

$$y = x^2$$
$$x = \pm\sqrt{y}$$
$$x_1 = \sqrt{y}\ ,\ x_2 = -\sqrt{y}$$

$$f_Y(y) = f_X(\sqrt{y})\cdot\frac{1}{2\sqrt{y}} + f_X(-\sqrt{y})\,\frac{1}{2\sqrt{y}}$$

$$f_Y(y) = \frac{1}{2\sqrt{y}}\left[f_X(\sqrt{y}) + f_X(-\sqrt{y})\right]$$



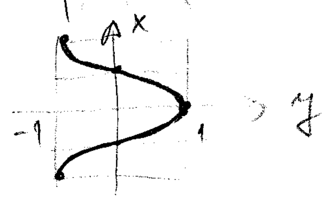
$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma_x}\, e^{-\frac{x^2}{2\sigma_x^2}}, \quad -\infty < x < \infty$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi y}\cdot\sigma_x}\, e^{-\frac{y}{2\sigma_x^2}}, \quad 0 < y < \infty$$

u opštem slučaju:
$$\boxed{f_Y(y) = \sum_k \frac{f_X(x_k)}{\left|\frac{dy}{dx_k}\right|}\ \Bigg|_{x_k = g^{-1}(y)}}$$

3 Odrediti raspodelu slučajne promenljive $Y = \cos X$ ako je X slučajna promenljiva sa uniformnom raspodelom u opsegu $(-\pi, \pi)$

Rezultat:



$$y = \cos x$$
$$x_1(y) = \arccos(y)$$
$$x_2(y) = -\arccos(y)$$

$$\Rightarrow f_Y(y) = \frac{1}{\pi\sqrt{1-y^2}}$$
za $-1 < y < 1$

$$\left|\frac{dx_1}{dy}\right| = \left|\frac{dx_2}{dy}\right| = \frac{1}{\sqrt{1-y^2}}$$

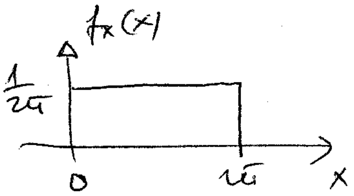
ili $\left|\frac{dy}{dx}\right| = |\sin x| = \sqrt{1-\cos^2 x} = \sqrt{1-y^2}$

$\boxed{\text{PDF}} \rightarrow \sin(\omega_1 t_0 + X) = Z$

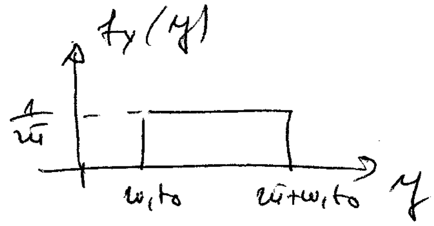
$\boxed{3}$

$$X = \text{Unif}(0, 2\pi)$$

1) $Y = \omega_1 t_0 + X = \text{Unif}(\omega_1 t_0, 2\pi + \omega_1 t_0)$

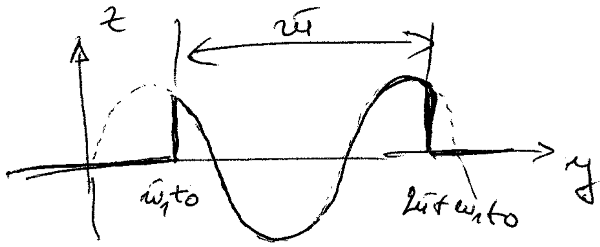


$\Rightarrow$



2) $Z = A\sin(Y)$

$z = A\sin(y)$

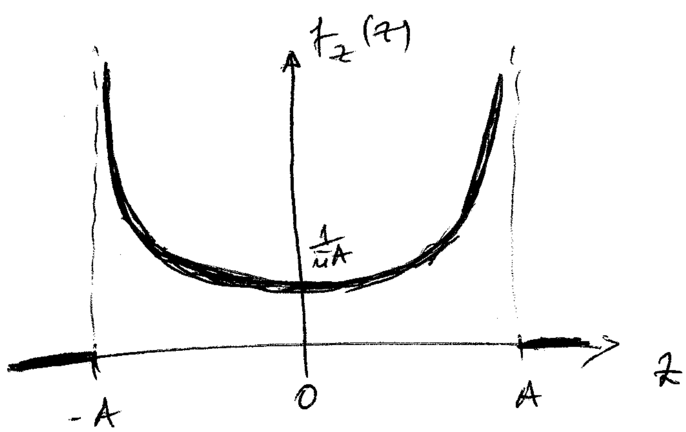


$$f_Z(z) = \frac{f_Y(y_1)}{\left|\frac{dz}{dy}\right|} + \frac{f_Y(y_2)}{\left|\frac{dz}{dy}\right|}, \quad z \in (-A, A)$$

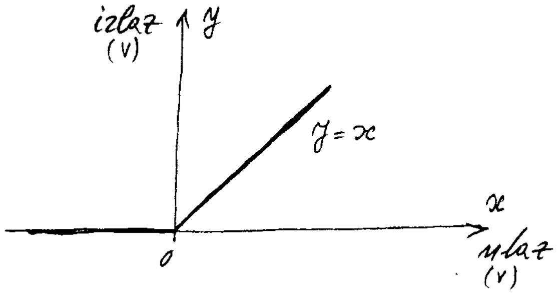
$$\left|\frac{dz}{dy}\right| = A|\cos(y)| = A\sqrt{1-\sin^2(y)} = \sqrt{A^2 - z^2}$$

$$f_Z(z) = \frac{\frac{1}{2\pi}}{\sqrt{A^2 - z^2}} \cdot 2, \quad z \in (-A, A)$$

$$f_Z(z) = \begin{cases} \frac{1}{\pi\sqrt{A^2 - z^2}}, & z \in (-A, A) \\ 0, & z \notin (-A, A) \end{cases}$$



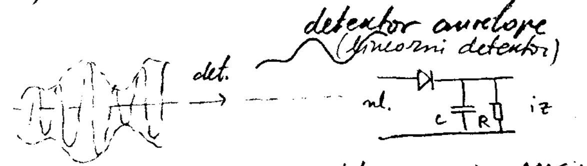
4 Na ulaz detektora, čija je funkcija prenosa prikazana na slici, deluje stacionarni slučajni proces $x(t)$ koji se pokorava normalnoj raspodeli i ima srednju vrednost $\bar{x}=0$ i varijansu σ^2. Potrebno je odrediti raspodelu gustine vatnoće slučajne promenljive $y(t)$ na izlazu detektora, srednju vrednost i varijansu tog procesa.



Veza procesa na ulazu i izlazu

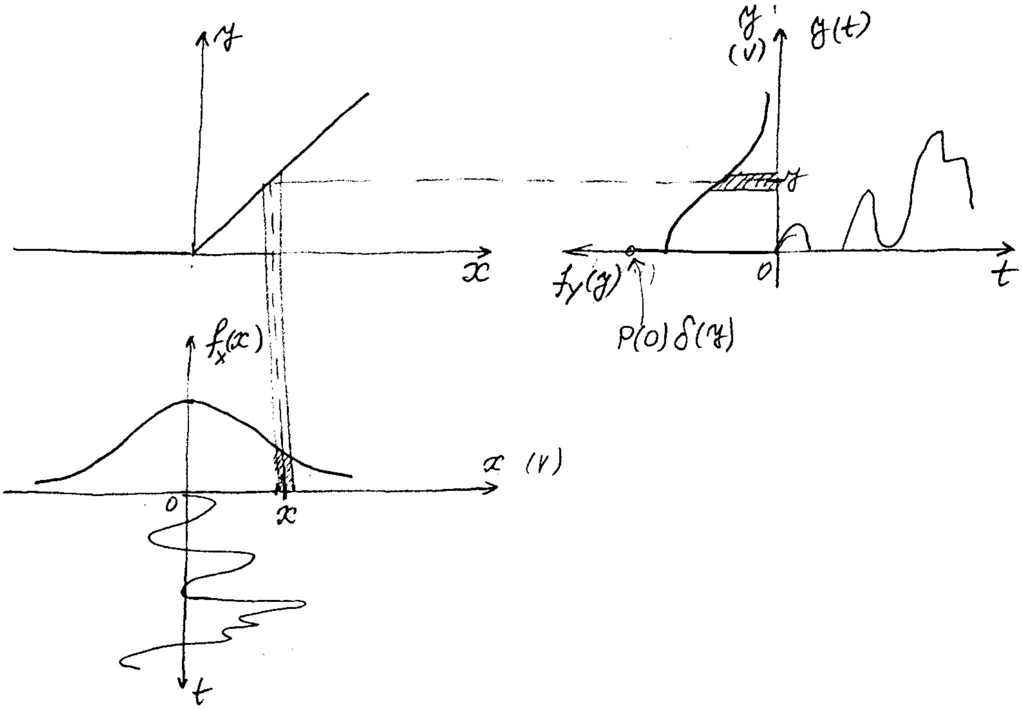
$y = x, \quad x > 0$

$y = 0, \quad x \leq 0$



det. konvencion AM sig.

Dopunimo sliku elementima koji će nam pomoći da rešimo zadatak



Gustina verovatnoće slučajne promenljive X je

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{x^2}{2\sigma^2}} \qquad -\infty < x < \infty \quad , \bar{x}=0$$

smena $u = \frac{x}{\sqrt{2}\sigma}$

$$P(X\leq 0) = \int_{-\infty}^{0} f_X(x)\,dx = \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{0} e^{-u^2}\sqrt{2}\sigma\,du = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{0} e^{-u^2}du = \frac{1}{2}$$

Gausov integral greške $\frac{1}{2}\{-\infty,0\} = \frac{1}{2}$

Verovatnoća da slučajni proces na izlazu Y uzme vrednost $y=0$ je takođe $P(y=0)=0.5$

Za $x>0$ gustina verovatnoće slučajne promenljive na izlazu je

$$f_Y(y) = \frac{f_X(x=f(y))}{\left|\frac{dy}{dx}\right|}$$

$$f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{y^2}{2\sigma^2}}, & y > 0 \\ \frac{1}{2}\delta(y), & y = 0 \\ 0, & y < 0 \end{cases}$$

$$\frac{y^2}{2\sigma^2} = u$$

$$\frac{\cancel{2}y\,dy}{\cancel{2}\sigma^2} = du$$

$$y\,dy = \sigma^2 du$$

$y = 0 \Rightarrow u = 0$

$y \to \infty \Rightarrow u \to \infty$

$$\bar{y} = \int_{-\infty}^{\infty} y f_Y(y)\,dy = \underbrace{\frac{1}{2}\int_{-\infty}^{+\infty} y\delta(y)\,dy}_{\to 0} + \int_0^{\infty} y \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y^2}{2\sigma^2}}\,dy$$

$$\bar{y} = \frac{\sigma^{\cancel{2}}}{\sqrt{2\pi}\,\cancel{\sigma}} \underbrace{\int_0^{\infty} e^{-u}\,du}_{1}$$

$$\int_{-\infty}^{\infty} f(x)\delta(x)\,dx = f(0)$$

$$\bar{y} = \frac{\sigma}{\sqrt{2\pi}}$$

$$\sigma_y^2 = \overline{(y-\bar{y})^2} = \dots = \overline{y^2} - \bar{y}^2$$

$$\overline{y^2} = \int_{-\infty}^{+\infty} y^2 f_Y(y)\,dy = \underbrace{\int_{-\infty}^{+\infty} y^2 \frac{1}{2}\delta(y)\,dy}_{\to 0} + \int_0^{\infty} y^2 \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y^2}{2\sigma^2}}\,dy$$

$$\overline{y^2} = \frac{1}{\sqrt{2\pi}\sigma} \int_0^{\infty} \underbrace{y}_{u}\, \underbrace{e^{-\frac{y^2}{2\sigma^2}}(y\,dy)}_{dv}$$

$$\boxed{\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du}$$

$$\boxed{u = y \Rightarrow du = dy}$$

$$dv = y e^{-\frac{y^2}{2\sigma^2}}\,dy$$

$$v = \int y e^{-\frac{y^2}{2\sigma^2}}\,dy \qquad \frac{y^2}{2\sigma^2} = t$$

$$v = \sigma^2 \int e^{-t}\,dt$$

$$v = -\sigma^2 e^{-t}$$

$$\boxed{v = -\sigma^2 e^{-\frac{y^2}{2\sigma^2}}}$$

$$\overline{y^2} = \frac{1}{\sqrt{2\pi}\sigma}\left(\underbrace{y\cdot(-\sigma^2) e^{-\frac{y^2}{2\sigma^2}}\Big|_0^{\infty}}_{\to 0} + \sigma^2 \int_0^{\infty} e^{-\frac{y^2}{2\sigma^2}}\,dy \right)$$

$$\overline{y^2} = \frac{\sigma}{\sqrt{2\pi}} \int_0^{\infty} e^{-\frac{y^2}{2\sigma^2}}\,dy \qquad u^2$$

$$\lim_{y\to\infty} y \cdot e^{-y^2}$$

$$\lim_{y\to\infty} \frac{y}{1 + \frac{y^2}{1!} + \frac{(y^2)^2}{2!} + \dots} = 0$$

$$\frac{y}{\sqrt{2}\sigma} = u \qquad y = 0 \Rightarrow u = 0$$

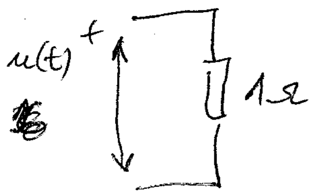
$$y \to \infty \Rightarrow u \to \infty$$

$$dy = \sqrt{2}\sigma\,du$$

$$\overline{y^2} = \frac{\sigma}{\cancel{\sqrt{2\pi}}} \cdot \sqrt{2}\sigma \frac{\sqrt{\pi}}{2} \underbrace{\frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-u^2}\,du}_{\mathrm{erf}(\infty) = 1} = \frac{\sigma^2}{2}$$

$$\sigma_Y^2 = \overline{y^2} - \overline{y}^2 = \frac{\sigma^2}{2} - \left(\frac{\sigma}{\sqrt{2\pi}}\right)^2 = \frac{\sigma^2}{2}\left(1 - \frac{1}{\pi}\right)$$

1. $\overline{x(t)}$ → srednja vrednost → jednosmerna komponenta
2. $(\overline{x(t)})^2$ → (-"-)^2 → jednosmerna snaga



$$p(t) = u(t) \cdot \underbrace{i(t)}_{\frac{u(t)}{R}} = \frac{u^2(t)}{R} = u^2(t)$$

R — 1Ω

$$\overline{p(t)} = \overline{u^2(t)}$$

$$\cancel{\overline{p(t)} = \overline{u^2(t)} = \overline{(U_{DC} + u_{AC}(t))^2} = \overline{U_{DC}^2 + 2U_{DC}u_{AC}(t) + u_{AC}^2(t)}}$$

$$\cancel{\overline{u^2(t)} = U_{DC}^2 + 2U_{DC}\overline{u_{AC}} + \overline{u_{AC}^2(t)}}$$

$$\cancel{\overline{u^2(t)} = U_{DC}^2 + \overline{u_{AC}^2(t)}}$$

$$\cancel{\overline{u^2(t)} = \overline{u(t)}^2 + \overline{u_{AC}^2(t)}}$$

$\overline{u^2(t)} = \overline{p(t)} = P_{DC} + P_{AC}$ — ukupna snaga

$P_{DC} = (\overline{u(t)})^2$ — DC snaga

$P_{AC} = \overline{p(t)} - P_{DC}$

$P_{AC} = \overline{u^2(t)} - \overline{u(t)}^2$ — AC snaga

$u(t)$ → sl. proc.

sr. vr. $(\overline{u(t)})$ → jednosmerne komponente

sr. kv. vr. (II moment) $\boxed{\overline{u^2(t)}}$ → ukupna snaga na $R = 1\Omega$

varijansa $\sigma_u^2 = \overline{u^2(t)} - \overline{u(t)}^2$ → AC snaga

(sr. vr.)^2 $\overline{u(t)}^2$ → DC snaga