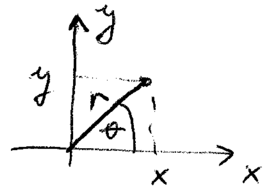
1 POSMATRAJU SE DVE NEZAVISNE SLUČAJNE PROMENLJIVE X i Y koje imaju Gausovu raspodelu srednje vrednosti $\emptyset$ i varijanse σ^2. ODREDITI RASPODELE SLUČAJNIH PROMENLJIVIH R i Θ KOJE SU DEFINISANE RELACIJAMA: $R=\sqrt{X^2+Y^2}$; $\Theta=\operatorname{arctg}\frac{Y}{X}$.

$$f_X(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{x^2}{2\sigma^2}}$$
$$f_Y(y)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{y^2}{2\sigma^2}}$$
$$\Rightarrow f_{XY}(x,y)=f_X(x)\cdot f_Y(y)$$
$$f_{XY}(x,y)=\frac{1}{2\pi\sigma^2}e^{-\frac{x^2+y^2}{2\sigma^2}}$$

$$\left.\begin{array}{l} r=\sqrt{x^2+y^2}=Q_1(x,y)\\ \theta=\operatorname{arctg}\frac{y}{x}=Q_2(x,y)\end{array}\right\}\Rightarrow \begin{array}{l} x=r\cos\theta=S_1(r,\theta)\\ y=r\sin\theta=S_2(r,\theta)\end{array}$$



$$f_{R\Theta}(r,\theta)=\left.\frac{f_{XY}(x,y)}{|J|}\right|_{\substack{x=S_1(r,\theta)\\ y=S_2(r,\theta)}}=\left.f_{XY}(x,y)\right|_{\substack{x=S_1(r,\theta)\\ y=S_2(r,\theta)}}\cdot|J'|$$

$$J=\begin{vmatrix}\frac{\partial Q_1}{\partial x} & \frac{\partial Q_1}{\partial y}\\ \frac{\partial Q_2}{\partial x} & \frac{\partial Q_2}{\partial y}\end{vmatrix}\qquad J'=\begin{vmatrix}\frac{\partial S_1}{\partial r} & \frac{\partial S_2}{\partial r}\\ \frac{\partial S_1}{\partial \theta} & \frac{\partial S_2}{\partial \theta}\end{vmatrix}=\begin{vmatrix}\frac{\partial S_1}{\partial r} & \frac{\partial S_1}{\partial \theta}\\ \frac{\partial S_2}{\partial r} & \frac{\partial S_2}{\partial \theta}\end{vmatrix}$$

SVEJEDNO JE JER $\det(A)=\det(A^T)$

Očigledno je lakše raditi sa J':

$$J'=\begin{vmatrix}\cos\theta & \sin\theta\\ -r\sin\theta & r\cos\theta\end{vmatrix}=r\cos^2\theta+r\sin^2\theta=r$$

$$f_{R\Theta}(r,\theta)=\frac{1}{2\pi\sigma^2}e^{-\frac{r^2\cos^2\theta+r^2\sin^2\theta}{2\sigma^2}}\cdot r=\frac{r}{2\pi\sigma^2}e^{-\frac{r^2}{2\sigma^2}};\quad r\geq 0,\ \theta\in(0,2\pi)$$

$$f_R(r)=\int_0^{2\pi}f_{R\Theta}(r,\theta)\,d\theta=\frac{r}{2\pi\sigma^2}e^{-\frac{r^2}{2\sigma^2}}\int_0^{2\pi}d\theta=\frac{r}{\sigma^2}e^{-r^2/2\sigma^2},\quad r\geq 0$$

$$f_\Theta(\theta)=\int_0^\infty f_{R\Theta}(r,\theta)\,dr=\frac{1}{2\pi\sigma^2}\int_0^\infty e^{-\frac{r^2}{2\sigma^2}}r\,dr\qquad\left(\frac{r^2}{2\sigma^2}=u\Rightarrow r\,dr=\sigma^2du\right)$$

$$f_\Theta(\theta)=\frac{\sigma^2}{2\pi\sigma^2}\underbrace{\int_0^\infty e^{-u}du}_{=1}=\frac{1}{2\pi},\quad\theta\in(0,2\pi)$$

② Posmatraju se dve nezavisne slučajne promenljive X i Y koje imaju Gausovu raspodelu srednje vrednosti 0 i varijanse σ^2. Odrediti raspodelu slučajne promenljive $Z = X+Y$.

I NAČIN (Jakobijan)

$$f_{XY}(x,y) = f_X(x) f_Y(y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$$

uvedemo dodatnu slučajnu promenljivu $W = Y$:

$$f_{ZW}(z,w) = \left.\frac{f_{XY}(x,y)}{|J|}\right|_{\substack{x=S_1(z,w)\\ y=S_2(z,w)}} = \frac{1}{2\pi\sigma^2} e^{-\frac{(z-w)^2+w^2}{2\sigma^2}}$$

$$\left.\begin{aligned} z &= Q_1(x,y) = x+y \\ w &= Q_2(x,y) = y \end{aligned}\right\} \Rightarrow \begin{aligned} x &= S_1(z,w) = z-w \\ y &= S_2(z,w) = w \end{aligned}$$

$$J = \begin{vmatrix} \frac{\partial Q_1}{\partial x} & \frac{\partial Q_1}{\partial y} \\ \frac{\partial Q_2}{\partial x} & \frac{\partial Q_2}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$$

$$f_{ZW}(z,w) = \frac{1}{2\pi\sigma^2} e^{-\frac{z^2-2zw+2w^2}{2\sigma^2}}$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_{ZW}(z,w)\,dw = \frac{1}{2\pi\sigma^2} \int_{-\infty}^{\infty} e^{-\frac{z^2-2zw+2w^2}{2\sigma^2}}\,dw$$

$$\left(\frac{1}{\sqrt{2}}z - w\sqrt{2}\right)^2 = \frac{1}{2}z^2 - 2zw + 2w^2$$

$$f_Z(z) = \frac{1}{2\pi\sigma^2} e^{-\frac{z^2}{4\sigma^2}} \cdot \int_{-\infty}^{\infty} e^{-\frac{\left(\frac{1}{\sqrt{2}}z - \sqrt{2}w\right)^2}{2\sigma^2}}\,dw$$

Smena: $\frac{\frac{1}{\sqrt{2}}z - \sqrt{2}w}{\sqrt{2}\sigma} = u$ $\quad dw = -\sigma\,du$

$w \to -\infty \Rightarrow u \to +\infty$

$w \to +\infty \Rightarrow u \to -\infty$

$$f_Z(z) = \frac{1}{2\pi\sigma} e^{-\frac{z^2}{4\sigma^2}} \int_{-\infty}^{\infty} e^{-u^2}\,du \qquad \int_{-\infty}^{\infty} e^{-u^2}\,du = \sqrt{\pi}$$

$$f_Z(z) = \frac{1}{\sqrt{2\pi(\sqrt{2}\sigma)^2}} e^{-\frac{z^2}{2\cdot(\sqrt{2}\sigma)^2}} \Rightarrow \text{GAUSOVA RASPODELA}: \quad \bar{Z} = 0, \quad Var(Z) = 2\sigma^2$$

II NAČIN: KONVOLUCIJA

$$|J| = 1 \Rightarrow f_{ZW}(z,w) = f_{XY}(x,y)\Big|_{\substack{x=z-w\\ y=w}} = f_X(x)\cdot f_Y(y)\Big|_{\substack{x=z-w\\ y=w}} = f_X(z-w)\cdot f_Y(w)$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_{ZW}(z,w)\,dw = \int_{-\infty}^{\infty} f_X(z-w)\, f_Y(w)\,dw = f_X(z) \circledast f_Y(z)$$

(NAPOMENA: Probati šta se dobije ako je: $z = aX + bY$, gde su a i b konstante)

$$f_Z(z) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(z-w)^2}{2\sigma^2}} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{w^2}{2\sigma^2}}\,dw$$

$$f_Z(z) = \frac{1}{2\pi\sigma^2} \int_{-\infty}^{\infty} e^{-\frac{(z-w)^2 + w^2}{2\sigma^2}}\,dw \Rightarrow \text{isto kao I NAČIN !}$$

III NAČIN: KARAKTERISTIČNE FUNKCIJE

$$F_X(j\Omega) = F_Y(j\Omega) = \mathcal{F}\left\{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}\right\}\Bigg|_{f=-\frac{\Omega}{2\pi}} = \frac{1}{\sqrt{2\pi\sigma^2}}\,\mathcal{F}\left\{e^{-\pi\left(\frac{x}{\sqrt{2\pi}\sigma}\right)^2}\right\}$$

$$= \cancel{\frac{\sqrt{2\pi}\sigma}{\sqrt{2\pi\sigma^2}}}\, e^{-\pi\left[\sqrt{2\pi}\sigma\cdot\left(-\frac{\Omega}{2\pi}\right)\right]^2} = e^{-\frac{(\sigma\Omega)^2}{2}}$$

$$F_Z(j\Omega) = F_X(j\Omega)\cdot F_Y(j\Omega) = F_X^2(j\Omega) = e^{-(\sigma\Omega)^2}$$

$$f_Z(z) = \mathcal{F}^{-1}\left\{e^{-(\sigma\Omega)^2}\right\}\Big|_{\Omega=-2\pi f} = \mathcal{F}^{-1}\left\{2\sqrt{\pi}\sigma\cdot e^{-\pi(2\sqrt{\pi}\sigma f)^2}\right\}\cdot\frac{1}{2\sqrt{\pi}\sigma}$$

(τ iz tablice)

$$f_Z(z) = \frac{1}{\sqrt{2\pi(\sqrt{2}\sigma)^2}}\, e^{-\frac{z^2}{2(\sqrt{2}\sigma)^2}}$$

(3) Posmatraju se dve nezavisne slučajne promenljive X i Y sa uniformnom raspodelom $X = U(-a, a)$ i $Y = U(-b, b)$. Odrediti raspodelu slučajne promenljive $Z = X+Y$ ako je

a) $a = b = 1$

b) $a = 1,\ b = 2$

a)

I) Jakobijan

II) konvolucija

III) karakteristične funkcija

$$F_X(j\Omega) = \frac{\sin(a\Omega)}{a\Omega}$$ (izvedeno ranije, a ima i u tablici)

$$F_Y(j\Omega) = \frac{\sin(b\Omega)}{b\Omega}$$

$$F_Z(j\Omega) = \frac{\sin(a\Omega)\sin(b\Omega)}{ab\,\Omega^2} \quad \left(= F_X(j\Omega)\cdot F_Y(j\Omega) \right)$$

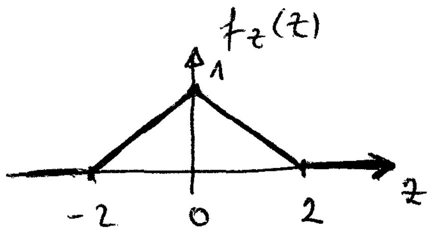
$$f_Z(z) = \mathcal{F}^{-1}\left\{F_Z(j\Omega)\right\} = \mathcal{F}^{-1}\left\{\frac{\sin(a\Omega)\sin(b\Omega)}{ab\,\Omega^2}\right\}\Bigg|_{\Omega=-2\pi f}$$

Rešavanje Furijeovog integrala za $a \neq b$ je dosta komplikovano, a nema gotovog rešenja u tablici !

za $a=b=1 \Rightarrow f_Z(z) = \mathcal{F}^{-1}\left\{\underbrace{\frac{\sin^2(\Omega)}{\Omega^2}}_{\tau=2}\right\}\Bigg|_{\Omega=-2\pi f}$ ← ovo postoji u TABLICI !!!

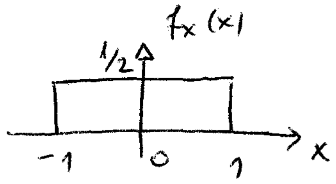
$$f_Z(z) = \mathcal{F}^{-1}\left\{\underbrace{1}_{B=1}\cdot\underbrace{2}_{\tau=2}\,\frac{\sin^2(\underbrace{2\pi f}_{\tau=2})}{(2\pi f)^2}\right\}\cdot\frac{1}{2}$$

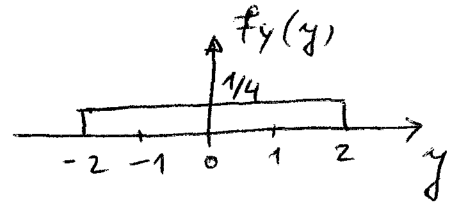
$$f_Z(z) = 1\cdot\Lambda\left(\frac{z}{2}\right)$$



PROVERA ! $\int_{-\infty}^{\infty} f_Z(z)\,dz = 1$

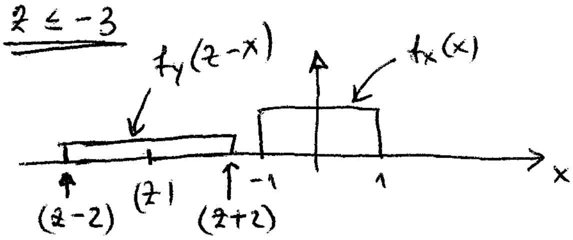
b)





I) KONVOLUCIJA: $f_Z(z) = f_X(z) \circledast f_Y(z) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(z-x)\, dx$

1) $z \leq -3$

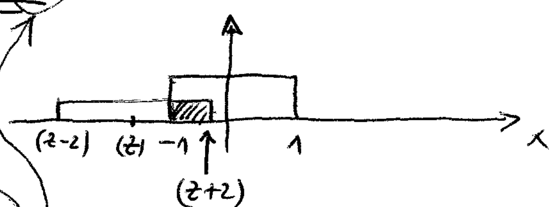


$f_Z(z) = 0$

2) $-3 \leq z \leq -1$

$z+2 = -1$
$z = -3$

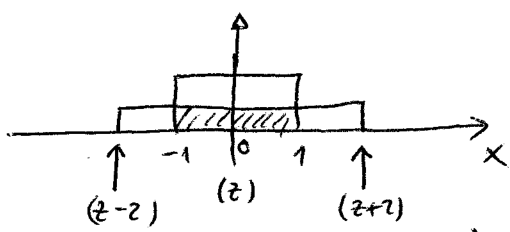
$z+2 = 1$
$z = -1$



$f_Z(z) = \int_{-1}^{z+2} \frac{1}{2} \cdot \frac{1}{4}\, dx = \frac{1}{8}(z+3)$

3) $-1 \leq z \leq 1$

$z-2 = -1$
$z = 1$

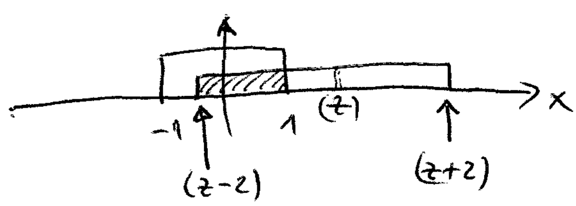


(uzeto je da je $z=0$)

$f_Z(z) = \int_{-1}^{1} \frac{1}{2} \cdot \frac{1}{4}\, dx = \frac{1}{4}$

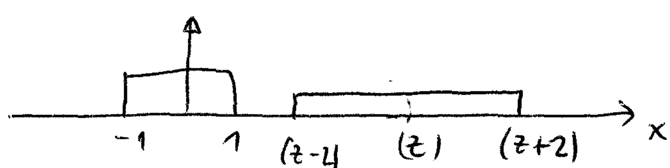
4) $1 \leq z \leq 3$

$z-2 = 1$
$z = 3$

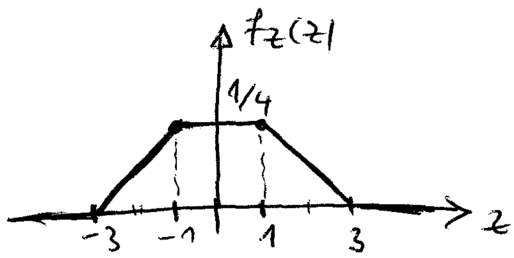


$f_Z(z) = \int_{z-2}^{1} \frac{1}{2} \cdot \frac{1}{4}\, dx = \frac{1}{8}(3-z)$

5) $z \geq 3$



$f_Z(z) = 0$



Provera: $\int_{-\infty}^{\infty} f_Z(z)\, dz = 1$

b) II način: Jakobijan

$$z = x + y = Q_1(x,y) \qquad x = z - w = S_1(z,w)$$
$$w = y = Q_2(x,y) \qquad y = w = S_2(z,w)$$

$$J = \begin{vmatrix} \partial Q_1/\partial x & \partial Q_1/\partial y \\ \partial Q_2/\partial x & \partial Q_2/\partial y \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$$

$$\left.\begin{aligned} f_x(x) &= \tfrac{1}{2}, \quad -1 < x < 1 \\ f_y(y) &= \tfrac{1}{4}, \quad -2 < y < 2 \end{aligned}\right\} \Rightarrow f_{xy}(x,y) = f_x(x) f_x(y) = \begin{cases} \tfrac{1}{8}; & -1<x<1,\ -2<y<2 \\ 0; & \text{inače} \end{cases}$$

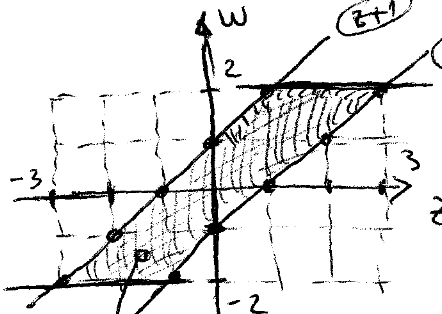
$$f_{zw}(z,w) = \frac{f_{xy}(x,y)}{|J|} = \begin{cases} \tfrac{1}{8}; & -1 < z-w < 1,\ -2 < w < 2 \\ 0; & \text{inače} \end{cases}$$

$-2 < w < 2$ → uslov 1

$$\boxed{\begin{array}{c} -1 < w - z < 1 \\ \hline z-1 < w < z+1 \end{array}} \Rightarrow \text{uslov } 2$$

$$f_z(z) = \int_{-\infty}^{\infty} f_{zw}(z,w)\, dw$$

INTEGRALI SE U GRANICAMA KOJE SU ODREĐENE PRESEKOM USLOVA 1 i 2 !

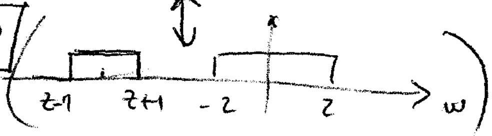


samo u ovom domenu je $f_{zw} = \frac{1}{8}$

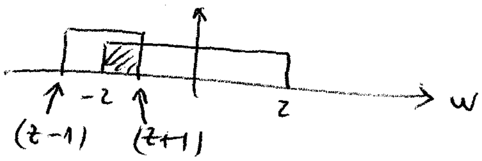
1) $z < -3$ ⇒ uslov 2: $-4 < w < -2$ (za $z = -3$)

⇒ nema preseka 1 i 2

$$\boxed{f_z(z) = 0}$$

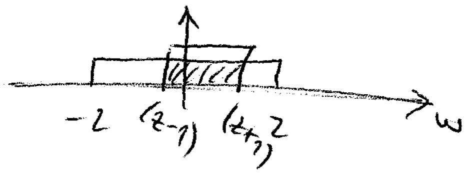


2) $-1 < z < 3$



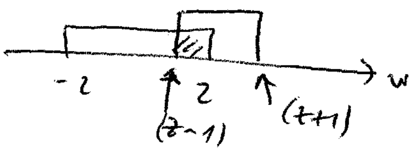
$$f_z(z) = \int_{-2}^{z+1} \frac{1}{8}\, dw = \frac{1}{8}(z+3)$$

3) $-1 < z < 1$



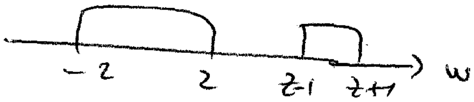
$$f_z(z) = \int_{z-1}^{z+1} \frac{1}{8}\, dw = \frac{1}{4}$$

4) $1 < z < 3$



$$f_z(z) = \int_{z-1}^{2} \frac{1}{8}\, dw = \frac{1}{8}(3-z)$$

5) $z > 3$



$$f_z(z) = 0$$