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Ојлерове смене и биномни диференцијал

1. (кол.) $$\int \frac{dx}{x-1+\sqrt{x^2-x+1}} = I$$

I начин $$\sqrt{x^2-x+1} = t-x \quad /^2$$

$$x^2-x+1 = t^2-2tx+x^2$$

$$x(2t-1) = t^2-1 \quad \Rightarrow \quad x = \frac{t^2-1}{2t-1}$$

$$dx = \frac{2t(2t-1)-(t^2-1)\cdot 2}{(2t-1)^2}\, dt = \frac{2t^2-2t+2}{(2t-1)^2}\, dt$$

$$I = \int \frac{\frac{2(t^2-t+1)dt}{(2t-1)^2}}{t-1} = \int \frac{2(t^2-t+1)dt}{(t-1)(2t-1)^2} = \int\left(\frac{A}{t-1}+\frac{B}{2t-1}+\frac{C}{(2t-1)^2}\right)dt$$

II начин $$\sqrt{x^2-x+1} = tx+1 \quad /^2$$

$$x^2-x+1 = t^2x^2+2tx+1$$

$$x-1 = t^2\cdot x+2t$$

$$x(1-t^2) = 2t+1 \quad \rightarrow \quad x = \frac{2t+1}{1-t^2}$$

$$dx = \frac{2(1-t^2)-(2t+1)\cdot(-2t)}{(1-t^2)^2}\, dt = \frac{2t^2+2t+2}{(1-t^2)^2}\, dt$$

$$I = \int \frac{\frac{2(t^2+t+1)}{(1-t^2)^2}\, dt}{\frac{2t+1}{1-t^2}+t\cdot\frac{2t+1}{1-t^2}} = \int \frac{\frac{2(t^2+t+1)}{(1-t^2)^2}}{\frac{2t^2+3t+1}{1-t^2}}\, dt =$$

$$= \int \frac{2(t^2+t+1)}{(1-t^2)\underbrace{(2t^2+3t+1)}_{(t+1)(2t+1)}}\, dt = \int \frac{2(t^2+t+1)}{(1-t)(1+t)^2(2t+1)}\, dt$$

$$= \int\left(\frac{A}{1-t}+\frac{B}{1+t}+\frac{C}{(1+t)^2}+\frac{D}{2t+1}\right)dt = \ldots$$

2. $\int \frac{dx}{x^2 \cdot \sqrt[3]{(1+x^3)^5}} = \int x^{-2} \cdot (1+x^3)^{-\frac{5}{3}} dx$

$m = -2, \quad n = 3, \quad p = -\frac{5}{3}$

$\frac{m+1}{n} = -\frac{1}{3} \notin \mathbb{Z}$, али $\frac{m+1}{n} + p = -2 \in \mathbb{Z}$, па је смена

$\frac{1}{x^3} + 1 = t^3 \Rightarrow x^{-3} = t^3 - 1$

$x = (t^3-1)^{-1/3}$

$dx = -\frac{1}{3}(t^3-1)^{-\frac{4}{3}} \cdot 3t^2 \, dt$

$$I = -\int (t^3-1)^{\frac{2}{3}} \left(1 + \frac{1}{t^3-1}\right)^{-\frac{5}{3}} \cdot t^2 \cdot (t^3-1)^{-4/3} \, dt =$$

$$= -\int t^2 \cdot (t^3-1)^{-\frac{2}{3}} \cdot \left(\frac{t^3}{(t^3-1)}\right)^{-\frac{5}{3}} dt = \int t^2 \cdot (t^3-1)^{-\frac{2}{3}} \cdot t^{-5} \cdot (t^3-1)^{\frac{5}{3}} dt$$

$$= \int \frac{t^3-1}{t^3} dt = \int (1 - t^{-3}) \, dt = t + \frac{t^{-2}}{2} + C$$ 34

$t = \left(\frac{1}{x^3} + 1\right)^{1/3}$

3. $\int \sqrt{\frac{x}{1 - x\sqrt{x}}} \, dx = \int x^{1/2} \cdot (1 - x^{3/2})^{-1/2} dx$

$m = \frac{1}{2}, \quad n = \frac{3}{2}, \quad p = -\frac{1}{2}$

$\frac{m+1}{n} = 1 \in \mathbb{Z}$, па је смена $1 - x^{3/2} = t^2$

$x^{3/2} = 1 - t^2 \Rightarrow x = (1-t^2)^{2/3}, \quad dx = \frac{2}{3}(1-t^2)^{-1/3} \cdot (-2t) \, dt$

$$I = \int \cancel{(1-t^2)^{1/3}} \cdot (t^2)^{-1/2} \cdot \left(\frac{-4}{3}\right) \cdot t \cdot \cancel{(1-t^2)^{-1/3}} \, dt =$$

$$= -\frac{4}{3} t + C = -\frac{4}{3} \sqrt{1 - x\sqrt{x}} + C$$